

# ON THE HADAMARD'S TYPE INEQUALITIES FOR L-LIPSCHITZIAN MAPPING

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ABSTRACT. In this paper, we establish some new inequalities of Hadamard's type for  $L$ -Lipschitzian mapping in two variables.

## 1. INTRODUCTION

Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex mapping defined on the interval  $I$  of real numbers and  $a, b \in I$ , with  $a < b$ . the following double inequality is well known in the literature as the Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

Let us now consider a bidimensional interval  $\Delta = [a, b] \times [c, d]$  in  $\mathbb{R}^2$  with  $a < b$  and  $c < d$ . A mapping  $f : \Delta \rightarrow \mathbb{R}$  is said to be convex on  $\Delta$  if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x, y) + (1-t)f(z, w)$$

holds, for all  $(x, y), (z, w) \in \Delta$  and  $t \in [0, 1]$ . A function  $f : \Delta \rightarrow \mathbb{R}$  is said to be on the co-ordinates on  $\Delta$  if the partial mappings  $f_y : [a, b] \rightarrow \mathbb{R}$ ,  $f_y(u) = f(u, y)$  and  $f_x : [c, d] \rightarrow \mathbb{R}$ ,  $f_x(v) = f(x, v)$  are convex where defined for all  $x \in [a, b]$  and  $y \in [c, d]$  (see [3]).

A formal definition for co-ordinated convex function may be stated as follows:

**Definition 1.** A function  $f : \Delta \rightarrow \mathbb{R}$  will be called co-ordinated convex on  $\Delta$ , for all  $t, s \in [0, 1]$  and  $(x, y), (u, w) \in \Delta$ , if the following inequality holds:

$$f(tx + (1-t)y, su + (1-s)w)$$

$$(1.1) \leq tsf(x, u) + s(1-t)f(y, u) + t(1-s)f(x, w) + (1-t)(1-s)f(y, w).$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex function which is not convex, (see, [3]). For several recent results concerning Hermite-Hadamard's inequality for some convex function on the co-ordinates on a rectangle from the plane  $\mathbb{R}^2$ , we refer the reader to ([1]-[3], [5], [6], [8], [9] and [11]).

Recently, in [3], Dragomir establish the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane  $\mathbb{R}^2$ .

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**Theorem 1.** Suppose that  $f : \Delta \rightarrow \mathbb{R}$  is co-ordinated convex on  $\Delta$ . Then one has the inequalities:

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
& \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\
& \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\
& \leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \\
& \quad \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\
& \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\end{aligned}$$

The above inequalities are sharp.

**Definition 2.** Consider a function  $f : V \rightarrow \mathbb{R}$  defined on a subset  $V$  of  $\mathbb{R}^n$ ,  $n \in \mathbb{N}$ . Let  $L = (L_1, L_2, \dots, L_n)$  where  $L_i \geq 0$ ,  $i = 1, 2, \dots, n$ . We say that  $f$  is  $L$ -Lipschitzian function if

$$|f(x) - f(y)| \leq \sum_{i=1}^n L_i |x_i - y_i|$$

for all  $x, y \in V$ .

The authors in [4] and [10] have proved the following inequalities of Hadamard's type for Lipschitzian mapping.

**Theorem 2.** Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be  $L$ -Lipschitzian on  $I$  and  $a, b \in I$  with  $a < b$ . Then, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{L(b-a)}{3}$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{L(b-a)}{4}.$$

For several recent results concerning Hadamard's type inequality for some  $L$ -Lipschitzian function, we refer the reader to ([4], [7], [10]).

The main purpose of this paper is to establish some Hadamard's type inequalities for  $L$ -Lipschitzian mapping in two variables.

## 2. HADAMARD'S TYPE INEQUALITIES

Firstly, we will start the proof of the Theorem 1 by using the definition of the co-ordinated convex functions as follows:

**Theorem 3.** *Suppose that  $f : \Delta \rightarrow \mathbb{R}$  is co-ordinated convex on  $\Delta$ . Then one has the inequalities:*

$$\begin{aligned}
 f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\
 (2.1) \qquad \qquad \qquad &\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
 \end{aligned}$$

*Proof.* According to (1.1) with  $x = t_1 a + (1 - t_1)b$ ,  $y = (1 - t_1)a + t_1 b$ ,  $u = s_1 c + (1 - s_1)d$ ,  $w = (1 - s_1)c + s_1 d$  and  $t = s = \frac{1}{2}$ , we find that

$$\begin{aligned}
 &f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
 &\leq \frac{1}{4} [f(t_1 a + (1 - t_1)b, s_1 c + (1 - s_1)d) + f((1 - t_1)a + t_1 b, s_1 c + (1 - s_1)d) \\
 &\quad + f(t_1 a + (1 - t_1)b, (1 - s_1)c + s_1 d) + f(t_1 a + (1 - t_1)b, (1 - s_1)c + s_1 d)].
 \end{aligned}$$

Thus, by integrating with respect to  $t_1, s_1$  on  $[0, 1] \times [0, 1]$ , we obtain

$$\begin{aligned}
 &f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
 &\leq \frac{1}{4} \left[ \int_0^1 \int_0^1 [f(ta + (1 - t)b, sc + (1 - s)d) + f(ta + (1 - t)b, (1 - s)c + sd)] ds dt \right. \\
 &\quad \left. + \int_0^1 \int_0^1 [f((1 - t)a + tb, sc + (1 - s)d) + f((1 - t)a + tb, (1 - s)c + sd)] ds dt \right].
 \end{aligned}$$

Using the change of the variable, we get

$$(2.2) \qquad f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx,$$

which the first inequality is proved. The proof of the second inequality follows by using (1.1) with  $x = a$ ,  $y = b$ ,  $u = c$  and  $w = d$ , and integrating with respect to

$t, s$  over  $[0, 1] \times [0, 1]$ ,

$$\begin{aligned}
& \int_0^1 \int_0^1 f(ta + (1-t)b, sc + (1-s)d) ds dt \\
& \leq \int_0^1 \int_0^1 [tsf(a, c) + s(1-t)f(b, c) + t(1-s)f(a, d) + (1-t)(1-s)f(b, d)] ds dt \\
& = \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.
\end{aligned}$$

Here, using the change of the variable  $x = ta + (1-t)b$  and  $y = sc + (1-s)d$  for  $s, t \in [0, 1]$ , we have

$$(2.3) \quad \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}.$$

We get the inequality (2.1) from (2.2) and (2.3). The proof is complete.  $\square$

**Theorem 4.** *Let  $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfy  $L$ -Lipschitzian conditions. That is, for  $(t_1, s_1)$  and  $(t_2, s_2)$  belong to  $\Delta := [a, b] \times [c, d]$ , then we have*

$$|f(t_1, s_1) - f(t_2, s_2)| \leq L_1 |t_1 - t_2| + L_2 |s_1 - s_2|$$

where  $L_1$  and  $L_2$  are positive constants. Then, we have the following inequalities:

$$(2.4) \quad \left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \right| \leq \frac{1}{16} (M_1 |b-a| + M_2 |d-c|)$$

$$\begin{aligned}
& \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \right| \\
(2.5) & \leq \frac{1}{12} (M_1 |b-a| + M_2 |d-c|)
\end{aligned}$$

where  $M_1 = [L_1 + L_3 + L_5 + L_7]$  and  $M_2 = [L_2 + L_4 + L_6 + L_8]$ .

*Proof.* Let  $t, s \in [0, 1]$ . Since  $ts + s(1-t) + t(1-s) + (1-t)(1-s) = 1$ , then we have

$$\begin{aligned}
& |tsf(a, c) + s(1-t)f(b, c) + t(1-s)f(a, d) + (1-t)(1-s)f(b, d) \\
& - f(ta + (1-t)b, sc + (1-s)d)| \\
(2.6) \quad & = |ts[f(a, c) - f(ta + (1-t)b, sc + (1-s)d)] \\
& + s(1-t)[f(b, c) - f(ta + (1-t)b, sc + (1-s)d)] \\
& + t(1-s)[f(a, d) - f(ta + (1-t)b, sc + (1-s)d)] \\
& + (1-t)(1-s)[f(b, d) - f(ta + (1-t)b, sc + (1-s)d)]| \\
& \leq ts[(1-t)L_1|b-a| + (1-s)L_2|d-c|] + s(1-t)[tL_3|b-a| + (1-s)L_4|d-c|] \\
& + t(1-s)[(1-t)L_5|b-a| + sL_6|d-c|] + (1-t)(1-s)[tL_7|b-a| + sL_8|d-c|] \\
& = (ts(1-t)[L_1 + L_3] + t(1-s)(1-t)[L_5 + L_7])|b-a| \\
& + (ts(1-s)[L_2 + L_6] + s(1-s)(1-t)[L_4 + L_8])|d-c|.
\end{aligned}$$

If we choose  $t = s = \frac{1}{2}$  in (2.6), we get

$$\begin{aligned}
(2.7) \quad & \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right| \\
& \leq \frac{1}{8}([L_1 + L_3 + L_5 + L_7]|b-a| + [L_2 + L_6 + L_4 + L_8]|d-c|).
\end{aligned}$$

Thus, if we put  $ta + (1-t)b$  instead of  $a$ ,  $(1-t)a + tb$  instead of  $b$ ,  $sc + (1-s)d$  instead of  $c$  and  $(1-s)c + sd$  instead of  $d$  in (2.7), respectively, then it follows that

$$\begin{aligned}
(2.8) \quad & \left| \frac{f(ta + (1-t)b, sc + (1-s)d) + f(ta + (1-t)b, (1-s)c + sd)}{4} \right. \\
& + \frac{f((1-t)a + tb, sc + (1-s)d) + f((1-t)a + tb, (1-s)c + sd)}{4} \\
& \left. - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right| \\
& \leq \frac{1}{8}([L_1 + L_3 + L_5 + L_7]|1-2t||b-a| + [L_2 + L_6 + L_4 + L_8]|1-2s||d-c|)
\end{aligned}$$

for all  $t, s \in [0, 1]$ . If we integrate the inequality (2.8) with respect to  $s, t$  on  $[0, 1] \times [0, 1]$

$$\begin{aligned}
& \left| \frac{1}{4} \int_0^1 \int_0^1 [f(ta + (1-t)b, sc + (1-s)d) + f(ta + (1-t)b, (1-s)c + sd)] dsdt \right. \\
& + \frac{1}{4} \int_0^1 \int_0^1 [f((1-t)a + tb, sc + (1-s)d) + f((1-t)a + tb, (1-s)c + sd)] dsdt \\
& \left. - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right| \\
& \leq \frac{1}{8} \left\{ [L_1 + L_3 + L_5 + L_7] |b-a| \int_0^1 \int_0^1 |1-2t| dsdt \right. \\
& \quad \left. + [L_2 + L_6 + L_4 + L_8] |d-c| \int_0^1 \int_0^1 |1-2s| dsdt \right\}.
\end{aligned}$$

Thus, using the change of the variable  $x = ta + (1-t)b$ ,  $y = (1-t)a + tb$ ,  $u = sc + (1-s)d$  and  $w = (1-s)c + sd$  for  $t, s \in [0, 1]$ , and

$$\int_0^1 \int_0^1 |1-2t| dsdt = \int_0^1 \int_0^1 |1-2s| dsdt = \frac{1}{2}$$

we obtain the inequality (2.4).

Note that, by the inequality (2.6), we write

$$\begin{aligned}
& |tsf(a, c) + s(1-t)f(b, c) + t(1-s)f(a, d) + (1-t)(1-s)f(b, d) \\
& - f(ta + (1-t)b, sc + (1-s)d)| \\
(2.9) \quad & \leq (ts(1-t)[L_1 + L_3] + t(1-s)(1-t)[L_5 + L_7])|b-a| \\
& + (ts(1-s)[L_2 + L_6] + s(1-s)(1-t)[L_4 + L_8])|d-c|.
\end{aligned}$$

for all  $t, s \in [0, 1]$ . If we integrate the inequality (2.9) with respect to  $s, t$  on  $[0, 1] \times [0, 1]$ , we have

$$\begin{aligned}
& \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dydx \right| \\
& \leq \frac{1}{12} ([L_1 + L_3 + L_5 + L_7] |b-a| + [L_2 + L_6 + L_4 + L_8] |d-c|)
\end{aligned}$$

and so we have the inequality (2.5), where we use the fact that

$$\int_0^1 \int_0^1 st(1-t)dsdt = \int_0^1 \int_0^1 s(1-s)(1-t)dsdt = \frac{1}{12}.$$

This completes the proof.  $\square$

### 3. THE MAPPING $H$

For a  $L$ -Lipschitzian function  $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ , we can define a mapping  $H : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  by

$$H(t, s) := \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, sy + (1-s)\frac{c+d}{2}\right) dydx.$$

Now, we give some properties of this mapping as follows:

**Theorem 5.** *Suppose that  $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  be  $L$ -Lipschitzian on  $\Delta := [a, b] \times [c, d]$ . Then:*

- (i) *The mapping  $H$  is  $L$ -Lipschitzian on  $[0, 1] \times [0, 1]$ .*
- (ii) *We have the following inequalities*

$$(3.1) \quad \left| H(t, s) - f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right| \leq \frac{L_1 t}{4} (b-a) + \frac{L_2 s}{4} (d-c)$$

$$(3.2) \quad \left| H(t, s) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dydx \right| \leq \frac{L_1(1-t)}{4} (b-a) + \frac{L_2(1-s)}{4} (d-c).$$

*Proof.* (i) Let  $t_1, t_2, s_1, s_2 \in [0, 1]$ . Then, we have

$$\begin{aligned}
& |H(t_2, s_2) - H(t_1, s_1)| \\
&= \frac{1}{(b-a)(d-c)} \left| \int_a^b \int_c^d f \left( t_2 x + (1-t_2) \frac{a+b}{2}, s_2 y + (1-s_2) \frac{c+d}{2} \right) dy dx \right. \\
&\quad \left. - \int_a^b \int_c^d f \left( t_1 x + (1-t_1) \frac{a+b}{2}, s_1 y + (1-s_1) \frac{c+d}{2} \right) dy dx \right| \\
&\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left| f \left( t_2 x + (1-t_2) \frac{a+b}{2}, s_2 y + (1-s_2) \frac{c+d}{2} \right) \right. \\
&\quad \left. - f \left( t_1 x + (1-t_1) \frac{a+b}{2}, s_1 y + (1-s_1) \frac{c+d}{2} \right) \right| dy dx \\
&= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left[ L_1 |t_2 - t_1| \left| x - \frac{a+b}{2} \right| + L_2 |s_2 - s_1| \left| y - \frac{c+d}{2} \right| \right] dy dx \\
&= \frac{L_1(b-a)}{4} |t_2 - t_1| + \frac{L_2(d-c)}{4} |s_2 - s_1|,
\end{aligned}$$

i.e., for all  $t_1, t_2, s_1, s_2 \in [0, 1]$ ,

$$(3.3) \quad |H(t_2, s_2) - H(t_1, s_1)| \leq \frac{L_1(b-a)}{4} |t_2 - t_1| + \frac{L_2(d-c)}{4} |s_2 - s_1|,$$

which yields that the mapping  $H$  is  $L$ -Lipschitzian on  $[0, 1] \times [0, 1]$ .

(ii) The inequalities (3.1) and (3.2) follow from (3.3) by choosing  $t_1 = 0$ ,  $t_2 = t$ ,  $s_1 = 0$ ,  $s_2 = s$  and  $t_1 = 1$ ,  $t_2 = t$ ,  $s_1 = 1$ ,  $s_2 = s$ , respectively.  $\square$

Another result which is connected in a sense with the inequality (2.5) is also given in the following:

**Theorem 6.** *Under the assumptions Theorem 5, then we get the following inequality*

$$\begin{aligned}
& \left| \frac{f \left( at + (1-t) \frac{a+b}{2}, cs + (1-s) \frac{c+d}{2} \right) + f \left( at + (1-t) \frac{a+b}{2}, ds + (1-s) \frac{c+d}{2} \right)}{4} \right. \\
& \quad \left. + \frac{f \left( bt + (1-t) \frac{a+b}{2}, cs + (1-s) \frac{c+d}{2} \right) + f \left( bt + (1-t) \frac{a+b}{2}, ds + (1-s) \frac{c+d}{2} \right)}{4} \right. \\
& \quad \left. - \frac{1}{(n_2 - n_1)(m_2 - m_1)} \int_{n_1}^{n_2} \int_{m_1}^{m_2} f(u, w) dw du \right| \\
&\leq \frac{1}{12} (M_1 |n_2 - n_1| t + M_2 |m_2 - m_1| s)
\end{aligned}$$

where  $M_1 = [L_1 + L_3 + L_5 + L_7]$  and  $M_2 = [L_2 + L_4 + L_6 + L_8]$ .



*Proof.* If we denote  $n_1 = at + (1-t)\frac{a+b}{2}$ ,  $n_2 = bt + (1-t)\frac{a+b}{2}$ ,  $m_1 = cs + (1-s)\frac{c+d}{2}$  and  $m_2 = ds + (1-s)\frac{c+d}{2}$ , then, we have

$$H(t, s) = \frac{1}{(n_2 - n_1)(m_2 - m_1)} \int_{n_1}^{n_2} \int_{m_1}^{m_2} f(u, w) dw du.$$

Now, using the inequality (2.5) applied for  $n_1, n_2, m_1$  and  $m_2$ , we have

$$\begin{aligned} & \left| \frac{f(n_1, m_1) + f(n_1, m_2) + f(n_2, m_1) + f(n_2, m_2)}{4} \right. \\ & \quad \left. - \frac{1}{(n_2 - n_1)(m_2 - m_1)} \int_{n_1}^{n_2} \int_{m_1}^{m_2} f(u, w) dw du \right| \\ & \leq \frac{1}{12} (M_1 |n_2 - n_1| + M_2 |m_2 - m_1|) \end{aligned}$$

from which we have the inequality (3.4). This completes the proof.  $\square$

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